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The Optimality of (S,s) Policies in the Dynamic Inventory Problem

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1. Summary

This paper considers the dynamic inventory problem with an ordering cost composed of a unit cost plus a reorder cost. It is shown that if the holding and shortage costs are linear, then the optimal policy in each period is always of the (S, s) type. More general conditions on the holding and shortage costs are given which imply the same result. A similar result is also given in the case of a time lag in delivery.

2. Introduction

An elaborate discussion of the history and general features of the inventory problem may be found in [2]. We shall content ourselves here with a brief description of the type of model introduced in [1] and discussed by a number of subsequent authors ([2], [3], [4]).

A sequence of purchasing decisions is made at the beginning of a number of regularly spaced intervals. These purchases contribute to a build-up of inventories which are then depleted by demands during the various intervals. We shall assume the demands to be independent observations from a common distribution function, though varying distributions may be treated by the same technique.

Various costs are charged during the successive periods, and the objective is to select the purchasing decisions so as to minimize the expectation of the discounted value of all costs. There are, generally speaking, three types of costs: a purchasing or ordering cost $c(z)$, where z is the amount purchased; a holding cost $h(\cdot)$, which is a function of the excess of supply over demand at the end of the period; and a shortage cost $p(\cdot)$, which is a function of the excess of demand over supply at the end of the period. Holding or shortage costs are charged at the end of every period, and ordering costs are charged when a purchase is made. We shall assume initially that purchases are made only at the beginning of the period and that delivery

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is instantaneous. In Section 4 the case of a time lag in delivery will be discussed.

If the stock level immediately after purchases are delivered is y, then the expected holding and shortage costs to be charged during that period are given by

$$L(y) = \begin{cases} \int_0^y h(y-\xi)\varphi(\xi)d\xi + \int_y^\infty p(\xi - y)\varphi(\xi)d\xi & y \geqq 0 , \\ \int_0^\infty p(\xi - y)\varphi(\xi)d\xi & y < 0 , \end{cases} \tag{1}$$

where φ is the density of the demand distribution.

Let us assume that the inventory problem has a horizon of n periods and that the problem is begun with an initial inventory of x units. Let $C_n(x)$ represent the expected value of the discounted costs during this n-period program if the provisioning is done optimally. (The discount factor will be denoted by α, and will be between 0 and 1.) Then it is easy to see that $C_n(x)$ satisfies the functional equation

$$C_n(x) = \min_{y \geqq x} \left\{ c(y-x) + L(y) + \alpha\int_0^\infty C_{n-1}(y-\xi)\varphi(\xi)\, d\xi \right\} , \tag{2}$$

and that if $y_n(x)$ is the minimizing value of y in (2), then $y_n(x) - x$ represents the optimal initial purchase. The purpose of this paper will be to show that under surprisingly weak conditions the optimal policy will be of a very simple type.

Let us begin by reviewing some of the work that has been done on the one-period problem ($n = 1$, and $C_0 \equiv 0$). The single-period problem is essentially a problem in the calculus and a considerable amount is known about it, in distinction to the sequential problem [2, chap. 8]. The simplest case is when the ordering cost is linear, i.e., $c(z) = c \cdot z$. In this case the optimal policy for the single-period model is frequently defined by a single critical number $\bar{x}$, as follows: If $x < \bar{x}$, buy $\bar{x} - x$, and if $x > \bar{x}$, do not buy. Analogous results frequently hold in the sequential problem, the optimal policy being defined by a sequence of critical numbers $\bar{x}_1, \bar{x}_2, \cdots$; see [3]. A sufficient condition for these results to hold is that $L(y)$ be convex, a condition which obtains when the holding and shortage costs are each convex increasing functions which vanish at the origin. A number of other sufficient conditions for the one-period model and the dynamic model are given by Karlin in [2, chaps. 8 and 9, respectively].

The situation is considerably more complex when the ordering cost is no longer linear. We shall concentrate on the simplest type of non-linear cost:

$$c(z) = \begin{cases} 0 & z = 0 , \\ K + c \cdot z & z > 0 . \end{cases} \tag{3}$$

K is usually described as the reorder cost.

With this type of ordering cost the optimal policy in the single-period model is frequently defined by a pair of critical numbers (S, s) as follows: If $x < s$, order $(S - x)$, and if $x > s$, do not order. There are examples in the single-period model in which such a policy is not optimal. However, if the holding and shortage costs are linear functions of their arguments [$h(u) = h \cdot u$ and $p(u) = p \cdot u$], or more generally if $L(y)$ is convex, then the optimal policy for the single-period model is (S, s) [2, chap. 8].

However, even with the assumption of linear holding and shortage costs, the literature is very meager on the properties of optimal policies for the dynamic model. Bratten has shown (see [2, chap. 9]) that if the density of demand is decreasing, the optimal policy for the dynamic model is defined by a sequence of pairs of critical numbers (S_1, s_1), (S_2, s_2), $\cdots$. The only other result is due to Karlin [2], viz.: if φ has a monotone likelihood ratio, if the holding and shortage costs are linear, and if $c + h > \alpha p$, then the optimal policy is of the same sort. Both of these results are rather restrictive, the former because it requires a decreasing density, and the latter because of the severe constraint on the costs.

In this paper we shall show that when the holding and shortage costs are linear, or more generally when $L(y)$ is convex, and the ordering cost is as described above, the optimal policy in the dynamic problem is *always* of the (S, s) type *without* any additional conditions.

The two results mentioned above are based on a study of the functions

$$G_n(y) = cy + L(y) + \alpha\int_0^\infty C_{n-1}(y - \xi)\varphi(\xi)\, d\xi\ . \tag{4}$$

It is optimal to order from x if and only if there is some y larger than x, with $G_n(x) > K + G_n(y)$; and if we do order from x, it is to that $y > x$ which minimizes $G_n(y)$. [See (2).] When either Bratten's condition or Karlin's condition is assumed, it may be shown that $G_n(y)$ decreases to a minimum and subsequently increases. If the minimizing value of y is denoted by S_n and if s_n is defined by

$$G_n(s_n) = G_n(S_n) + K\ , \tag{5}$$

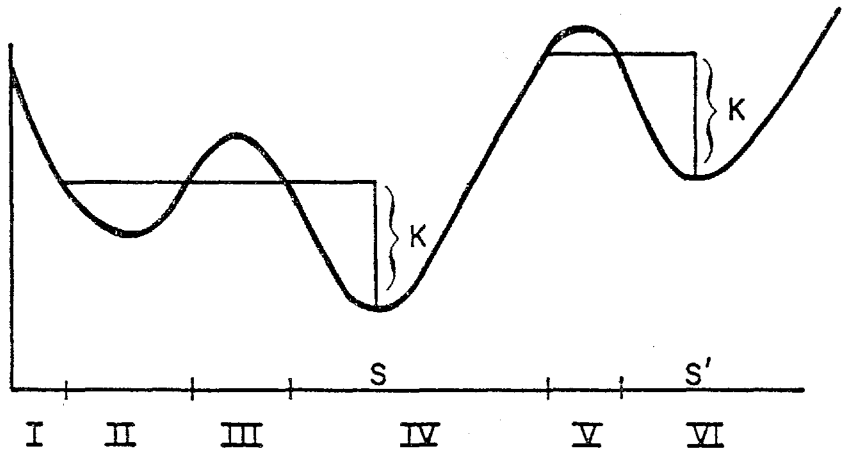
then the policy defined by (S_n, s_n) is indeed optimal. However, a few numerical calculations are sufficient to show that the functions G_n do not always have this regular behavior; they may actually have a number of maxima and minima. The idea of the proof given in this paper is that although G_n may have a large number of maxima and minima, the oscillations are never sufficiently large to cause a deviation from the (S, s) policy.

Explicitly, what we shall demonstrate is that if $L(y)$ is convex, the following inequality holds: *Let* $a \geqq 0$; *then*

$$K + G_n(a + x) - G_n(x) - aG_n'(x) \geqq 0\ . \tag{6}$$

To see that (6) implies that the optimal policy is (S, s), let us examine the accompanying graph of $G_n(x)$, which illustrates a typical case in which more complex policies are to be expected. With this type of graph for G_n, we would order in interval I to the point S, not order in interval II, order in

III to S, not order in IV, order in V to S' and not order in VI. But if (6) is correct, this sort of graph is impossible; for let $x + a = S$ and x be the point in III at which the relative maximum is attained. For this value of x, $G_n'(x) = 0$, and (6) implies that $K + G_n(S) - G_n(x) \geqq 0$, which contradicts the graph. The same argument may be applied to the point S'.



3. The Case of Zero Time Lag

In this section we consider the case in which delivery of orders is instantaneous. It will be shown that if $L(x)$ is convex and the ordering costs are given by (3), the optimal policies are of the (S, s) type.

In order to demonstrate (6) we shall make use of the following definition:

DEFINITION. *Let $K \geqq 0$, and let $f(x)$ be a differentiable function. We say that $f(x)$ is K-convex if*

(7) $$K + f(a + x) - f(x) - af'(x) \geqq 0, \text{ for all positive } a \text{ and all } x.$$

If differentiability is not assumed, then the appropriate definition of K-convexity would be

(8) $$K + f(a + x) - f(x) - a\left[\frac{f(x) - f(x - b)}{b}\right] \geqq 0.$$

Inasmuch as our applications will be to differentiable functions, we shall use (7) rather than (8). It may be shown that (7) implies (8), and of course (8) implies (7) if $f(x)$ is differentiable.

There are a number of simple properties of K-convex functions which will be of some use to us:

(i) 0-convexity is equivalent to ordinary convexity.

(ii) If $f(x)$ is K-convex, then $f(x + h)$ is K-convex for all h.

(iii) If f and g are K-convex and M-convex, respectively, then $\alpha f + \beta g$ is $(\alpha K + \beta M)$-convex when α and β are positive. This property may be extended to denumerable sums and integrals whenever the interchange of limits is permissible.

Now let us turn our attention to a proof of (6). We shall show inductively that each of the functions $G_1(x)$, $G_2(x)$, $\cdots$ are K-convex. G_1 is clearly K-convex, since $G_1(x)$ equals $cx + L(x)$, which is 0-convex and therefore K-convex. Let us assume that $G_1, \cdots, G_n$ are K-convex. If we examine (4), we see that in order to demonstrate the K-convexity of $G_{n+1}(x)$, it is sufficient to show that

$$\int_0^\infty C_n(x-\xi)\varphi(\xi)\,d\xi$$

is K-convex, and by properties (ii) and (iii) above, it is sufficient to show that $C_n(x)$ is K-convex.

The K-convexity of $C_n(x)$ may be shown as follows. We first notice that the argument of Section 2 demonstrates, as a consequence of the K-convexity of $G_n(x)$, that the optimal policy for the n-period problem is (S, s). In other words, if S_n is the absolute minimum of $G_n(x)$, and if s_n is defined as the value of $x < S_n$ satisfying $K + G_n(S_n) = G_n(s_n)$, then the optimal policy is to order to S_n if $x < s_n$ and otherwise not to order. Therefore

$$C_n(x) = \begin{cases} K + c(S_n - x) + C_n(S_n) = K - cx + G_n(S_n) & x < s_n\,, \\ -cx + G_n(x) & x > s_n\,. \end{cases} \tag{9}$$

We shall use (9) to demonstrate the K-convexity of $C_n(x)$. We distinguish three cases, using the notation of (7).

Case 1. $x > s_n$.

In this region $C_n(x)$ is equal to a linear function plus a K-convex function and is therefore K-convex.

Case 2. $x < s_n < x + a$.

In this case

$$K + C_n(x+a) - C_n(x) - aC_n'(x) = K + C_n(x+a) - C_n(x) + ac,$$

and this is positive since

$$\begin{aligned} C_n(x) &= \min_{y>x}\left\{K + c(y-x) + L(y) + \alpha\int_0^\infty C_{n-1}(y-\xi)\varphi(\xi)\,d\xi\right\} \\ &\leqq K + ca + L(x+a) + \alpha\int_0^\infty C_{n-1}(x+a-\xi)\varphi(\xi)\,d\xi \\ &= K + ca + C_n(x+a). \end{aligned}$$

(We use the fact that $x + a > s_n$, and therefore it is optimal not to order from $x + a$.)

Case 3. $x + a < s_n$.

In this region $C_n(x)$ is linear and therefore K-convex. This completes the induction, and demonstrates the optimality of (S, s) policies for the case considered in this section.

4. The Case of a Time Lag in Delivery

When there is a time lag in delivery, the character of optimal policies is very much dependent upon whether excess demand is backlogged or expedited; see [2, chap. 10]. If excess demand is backlogged, it is known that the optimal policy is a function of stock on hand plus stock ordered but not yet delivered, whereas if excess demand is expedited, the optimal policy never has this simple form. We shall restrict ourselves to the backlog case.

Let the time lag be denoted by λ, so that an order placed at the beginning of a period is delivered λ periods later at the beginning of the period. Consider a problem with a horizon of n periods. Let x represent current stock, x_1 stock to be delivered at the beginning of the next period, and generally speaking, x_j stock to be delivered j periods later, where $j = 1, 2, \cdots, \lambda - 1$. Let $C_n(x, x_1, \cdots, x_{\lambda-1})$ be the minimum expected cost for such a program. Then it is easy to see that this function satisfies an equation analogous to (2), namely

$$
(10) \qquad C_n(x, x_1, \cdots, x_{\lambda-1}) = \min_{z \geq 0} \left\{ c(z) + L(x) + \alpha \int_0^\infty C_{n-1}(x + x_1 - \xi, x_2, \cdots, z)\varphi(\xi)\, d\xi \right\},
$$

and that the minimizing value of z in this equation represents the optimal purchase.

We shall next demonstrate that if $L(x)$ is convex and the purchase costs are given by (3), the optimal policy is described by two numbers S_n and s_n as follows: If $x + x_1 + \cdots + x_{\lambda-1} > s_n$, do not order; if $x + x_1 + \cdots + x_{\lambda-1} < s_n$, order up to S_n.

The proof begins with a repetition of the argument in [2, p. 159]. It follows from (10) that C_n may be written in the following form (for $n \geqq \lambda$):

$$
\begin{aligned}
(11) \qquad & C_n(x, x_1, x_2, \cdots, x_{\lambda-1}) \\
&= L(x) + \alpha \int_0^\infty L(x + x_1 - \xi)\varphi(\xi)\, d\xi + \cdots \\
&\quad + \alpha^{\lambda-1} \int_0^\infty \cdots \int_0^\infty L\left(x + \cdots + x_{\lambda-1} - \sum_{i=1}^{\lambda-1} \xi_i\right)\varphi(\xi_1) \cdots \varphi(\xi_{\lambda-1})\, d\xi_1 \cdots d\xi_{\lambda-1} \\
&\quad + f_n(x + x_1 + \cdots + x_{\lambda-1}),
\end{aligned}
$$

where $f_n(u)$ satisfies the functional equation

$$
\begin{aligned}
(12) \quad f_n(u) = \min_{z \geq 0} \Bigg\{ & c(z) + \alpha^\lambda \int_0^\infty \cdots \int_0^\infty L\left(u + z - \sum_{i=1}^{\lambda} \xi_i\right)\varphi(\xi_1) \cdots \varphi(\xi_\lambda)\, d\xi_1 \cdots d\xi_\lambda \\
& + \alpha \int_0^\infty f_{n-1}(u + z - \xi)\varphi(\xi)\, d\xi \Bigg\}.
\end{aligned}
$$

It follows also from (10) that the minimizing value of z gives the optimal purchase if

$$x + \sum_{j=1}^{\lambda-1} x_j = u.$$

(The initial conditions are $f_1(u) = \cdots = f_\lambda(u) = 0$.) If we write $y = u + z$, then (12) is identical with (2), except for the fact that $L(y)$ has been replaced by

$$\alpha^\lambda \int_0^\infty \cdots \int_0^\infty L\left(y - \sum_{i=1}^{\lambda} \xi_i\right) \varphi(\xi_1) \cdots \varphi(\xi_\lambda)\, d\xi_1 \cdots d\xi_\lambda.$$

However, if $L(y)$ is convex, then its replacement is also convex, and this is all that is necessary to repeat the argument of Section 3. This concludes the proof of the optimality of (S, s) policies in the time-lag case.

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