Cowles Foundation Paper #655

CHAPTER 6

Testing for optimality in the absence of convexity

Herbert E. Scarf

1 The fundamental theorems of welfare economics

The modern treatment of the fundamental theorems of welfare economics was developed by two of the masters of our trade: Kenneth J. Arrow (1951) and Gerard Debreu (1951). The contrast between their presentation and that offered by a distinguished predecessor – Abba Lerner – is a striking illustration of the emergence of a new line of argumentation in economic theory. The index to Lerner's book, *The economics of control* (1944), contains not a single reference to convex sets nor to the separating hyperplane theorem: The basic mathematical tools used by Arrow and Debreu to demonstrate the relationship between prices and Pareto optimality.

The first major theorem of welfare economics states that a competitive equilibrium is a Pareto-optimal production and distribution plan. The proof offered by Arrow is astonishingly brief; a line or two of mathematical argument replaces the tedious evaluation of vast arrays of marginal productivities and marginal rates of substitution. Moreover, convexity assumptions are required neither on the consumption nor production sides of the economy, though, of course, in the absence of such assumptions, the theorem is, in general, vacuous. The second major theorem – that a Pareto-optimal production and distribution plan can be supported by competitive prices – does require that consumer preferences satisfy a convexity assumption and that production sets be convex. Its proof is then a simple exercise in the application of Minkowski's separating hyperplane theorem.

The separating hyperplane theorem was in the air during the late 1940s. It had been used to provide an elementary proof of von Neumann's minimax theorem for two-person zero-sum games; and as Arrow remarks in his collected papers (1983), he had been present at a lecture given by Albert Tucker in which a variant of the separating hyperplane theorem was used

Reprinted from
Social choice and public decision making
Essays in honor of Kenneth J. Arrow, Volume I
Edited by Walter P. Heller, Ross M Starr and David A. Starrett

to demonstrate the existence of supporting prices for the general convex programming problem. The Kuhn–Tucker theorem can, of course, be viewed as a special case of the second welfare theorem in which there is a single consumer with a utility function depending only on the good whose output is being maximized. A further specialization to the linear activity analysis model of production leads directly to the economically significant aspect of the duality theorem for linear programming: the existence of prices that yield a zero profit for those activities in use at the optimum and a nonpositive profit for the remaining activities.

The general convex programming problem can be put in the form

$$\begin{aligned} &\max f_0(x_1, \ldots, x_n) \\ \text{subject to } &f_1(x_1, \ldots, x_n) \leq b_1, \\ &\vdots \\ &f_m(x_1, \ldots, x_n) \leq b_m, \\ &x \geq 0, \end{aligned}$$

and the appropriate convexity requirement is embodied in the assumption that f_0 is a concave function and that $f_1, \ldots, f_m$ are convex functions of their arguments. Assuming that the constraints are consistent, that the problem has a finite maximum at x^*, and that a mild regularity condition known as the constraint qualification holds, the second welfare theorem asserts the existence of nonnegative prices $\pi_1^*, \ldots, \pi_m^*$ with the property that

$$f_0(x^*) - \sum_1^m \pi_i^* b_i \geq f_0(x) - \sum_1^m \pi_i^* f_i(x) \quad \text{for all } x \geq 0.$$

The first welfare theorem may be interpreted in this context by saying that a feasible vector of activity levels x^* is the constrained maximum if there exists a nonnegative vector of prices π^*, one for each constraint, such that profits are maximized at x^*. A test based on prices is sufficient to verify that a proposed feasible solution is optimal, and under the assumption of convexity, such a pricing test is always available.

With the additional assumption that $f_0(x)$ is *strictly* concave, and that each $f_i(x)$, for $i = 1, \ldots, m$, is *strictly* convex, a numerical algorithm based on the Walrasian *tâtonnement,* may be shown to converge globally to the correct vector of prices π^*. Let π be an arbitrary nonnegative price vector, and let $x(\pi)$ maximize

$$f_0(x) - \sum_1^m \pi_i f_i(x) \quad \text{for all } x \geq 0.$$

The excess demand for the ith factor of production is then

$$f_i(x(\pi)) - b_i.$$

Our economic intuition that the price of a factor in excess demand will rise, and that the price of a factor whose demand is less than its supply will fall, may be captured by the system of nonlinear differential equations

$$d\pi_i/dt = f_i(x(\pi)) - b_i.$$

Given an arbitrary initial price vector π, the system of differential equations will typically have a solution $\pi_i(t)$ with $\pi_i(0) = \pi_i$. In the interest of simplicity, let us assume that $\pi_i(t) > 0$ for all t and that $\pi_i^* > 0$; otherwise, the differential equations must be modified so as to ensure that none of the prices become negative.

This system of differential equations is formally identical to the system introduced by Samuelson (1941–2) and studied extensively by Arrow and Hurwicz (1958) and Arrow, Block, and Hurwicz (1959) in the more general context in which consumers are explicitly incorporated in the model. I was first made aware of the problem of global stability of the price adjustment mechanism by Arrow himself during the close personal and professional association that I was privileged to enjoy in the late 1950s at Stanford.

With the considerable hindsight offered by several decades of continued research we now know that the Walrasian *tâtonnement* need not be globally stable; a model of exchange may easily be constructed so that the solution to the system of differential equations traces out any path on the nonnegative part of the unit sphere. But these complexities are related to the presence of consumers in the model; in the special case of convex programming, the price adjustment mechanism may be shown to be globally stable by verifying that $\sum_i (\pi_i(t) - \pi_i^*)^2$ decreases to zero along the solution path.

The pricing test for optimality is also available for the general linear programming problem:

$$\begin{aligned} \max a_{01}x_1 + \cdots + a_{0n}x_n&, \\ a_{11}x_1 + \cdots + a_{1n}x_n &\geq b_1, \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &\geq b_m, \\ x &\geq 0. \end{aligned}$$

A feasible solution to the system of inequalities is optimal if a vector of prices can be found such that each activity in use makes a profit of zero and the remaining activities make a nonpositive profit. And in a spirit that is similar to the Walrasian price adjustment mechanism, the search for economically significant prices is at the heart of that workhorse of linear programming: the simplex method. In the simplex method, a feasible solution to the system of linear inequalities is proposed, and prices are found so as to yield a profit of zero for those activities being used. The feasible solution is optimal if none of the remaining activities make a positive profit; otherwise, we select one of the profitable activities and use it at a positive level, making compensating changes in the activity levels previously specified. When one of those activity levels drops to zero, the pricing test is repeated, and optimality is reached in a finite number of iterations.

The convexity assumption is the mortar that binds together this remarkable edifice of existence theorems and computational algorithms. Unfortunately, convexity of the production set is not a strikingly realistic description of economic reality. Convexity requires that the production possibility set exhibit constant or decreasing returns to scale: That you or I can manufacture automobiles in our own backyards with the same degree of efficiency as that achieved by the Ford Motor Co. Economies of scale based on large indivisible pieces of machinery or forms of productive organization such as the assembly line, which are not economically merited at small scales of operation, are a major ingredient of the industrial revolution of the last 100 years. And their workings cannot be captured, either theoretically or computationally, by the competitive paradigm.

The following quotation shows most clearly Lerner's appreciation of the incompatibility between indivisibilities and competitive markets: "We see then that indivisibility leads to an expansion in the output of the firm, and this either makes the output big enough to render the indivisibility insignificant, or it destroys the perfection of competition. Significant indivisibility destroys perfect competition" (Lerner 1944, p. 176).

2 Neighborhood systems for production sets with indivisibilities

The most extreme example of a production possibility set involving indivisibilities is that described by an activity analysis model in which the activity levels are required to assume integral values. The mathematical programming problems that then arise by specifying a particular factor endowment are known as *integer programming problems* and have the form

$$\begin{aligned}\max\ & a_{01}h_1+\cdots+a_{0n}h_n,\\ & a_{11}h_1+\cdots+a_{1n}h_n\geq b_1,\\ & \qquad\vdots\\ & a_{m1}h_1+\cdots+a_{mn}h_n\geq b_m,\\ & \text{and } h_1,\ldots,h_n \text{ integral.}\end{aligned}$$

In contrast to linear programming, the optimal solution to an integer programming problem need not be supported by competitive prices. Consider, for example, the integer program

$$\begin{aligned}\max\ & -4h_1-3h_2,\\ & 2h_1+h_2\geq 3,\\ & h_1\geq 0,\\ & h_2\geq 0,\end{aligned}$$

and h_1, h_2 integral. The constraint set is drawn in Figure 1, with the objective function indicated by a dashed line.

When the integrality assumption is relaxed, the optimal solution to the corresponding linear program is to use only the first activity at the level of $\frac{3}{2}$. If the prices associated with the objective function and the first constraint are in a 1:2 ratio, the first activity will make a profit of zero and the second activity a negative profit. A decentralized profit maximizing response will lead to the selection of activities that are consistent with the optimal solution to the programming problem. On the other hand, when the variables are required to be integral, the optimal solution is at the point $(1, 1)$. Both activities are used at the optimal solution, and there is no price ratio yielding a zero profit for the two activities simultaneously.

The phenomenon illustrated by this example is a general one: Indivisibilities in production are incompatible with competitive factor markets. Prices are not available to verify that a feasible solution to an integer programming problem is actually optimal, and no computational algorithm based on prices can be successful in general. Of course, the subject of integer programming is not a new one, and there are many algorithms that perform quite well in practice, but I am unaware of any computational procedure the steps of whose execution are capable of the most rudimentary economic interpretation in terms of prices.

What I have proposed in a series of publications (Scarf 1981; in press) is the replacement of the neoclassical pricing test for optimality by a quantity test; more specifically, by a search through neighbors of a proposed feasible solution to see whether a nearby vector of activity levels is also

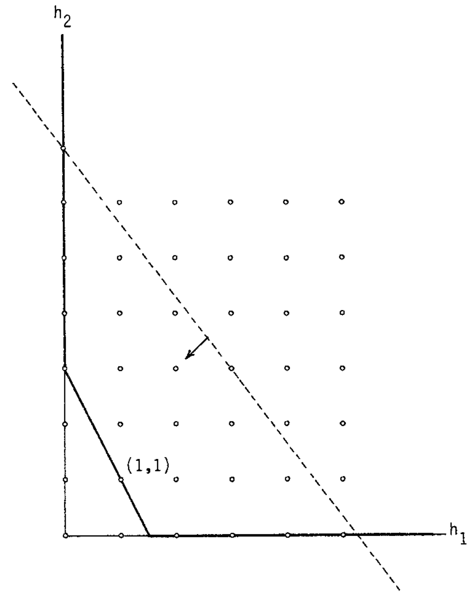

Figure 1. An example.

feasible and yields an improvement in the objective function. Consider the general integer program with n variables, ranging over all lattice points in n-dimensional Euclidean space. By a *neighborhood system,* I mean the association with each lattice point h of a *finite* set of neighbors $N(h)$; the association is arbitrary aside from the following two requirements:

1. if $k \in N(h)$, then $h \in N(k)$ and
2. $N(h) = h + N(0)$.

The first assumption states that the neighborhood relation is a symmetric one and the second that the neighbors of any two lattice points are translates of each other.

Given an integer program described by the technology matrix

$$A = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

and a specification of the right-hand side $b=(b_1, b_2, \dots, b_m)$, an arbitrary neighborhood system can be used to define the concept of a *local maximum:* a feasible vector of activity levels h all of whose neighbors $k \in N(h)$ are either infeasible or yield an inferior value of the objective function.

In general, a local maximum with respect to a particular neighborhood system need not be a global maximum to the programming problem; a particular neighborhood of a feasible point may simply omit some vector of activity levels that is also feasible and that improves the objective function. Moreover, such an improvement need not take place at a lattice point that is close to the original feasible vector in the sense of Euclidean distance. To see this, we merely remark that integer programs based on the technology matrices A and AU, with U a *unimodular* matrix (a nonsingular matrix with integer entries and a determinant of ± 1) are equivalent: If h is the optimal solution to an integer program with technology matrix A, then $h' = U^{-1}h$ is the optimal solution to the corresponding problem with matrix AU. But unimodular transformations do not preserve Euclidean distance, and nearby points may be mapped into points that are quite far apart.

The following theorem may be demonstrated quite easily (Scarf, 1984).

Theorem 2.1. Under mild regularity assumptions on the technology matrix A, there exists a *unique, minimal* neighborhood system such that a local maximum is global for all integer programs based on A. This neighborhood system depends on the technology matrix alone and not on the specification of the particular factor endowment.

The regularity assumptions referred to in the statement of Theorem 2.1 may be stated as follows:

Assumption 2.2. For each $b=(b_0, b_1, \dots, b_m)$, the set of integral vectors h such that $Ah \geq b$ is finite. Moreover, the entries in each row of A are independent over the integers in the sense that $\sum_j a_{ij} h_j = 0$ for *any* i implies that $h=0$.

To demonstrate Theorem 2.1, we argue that two lattice points h and h' must be neighbors - in any neighborhood system for which a local maximum is global - if there is some vector $b=(b_0, b_1, \dots, b_m)$ such that the only lattice points satisfying $Ah \geq b$ are h and h' themselves. For let b be such a vector and assume that $\sum_j a_{0j} h'_j > \sum_j a_{0j} h_j$. It follows that h is a feasible solution to the integer program

$$\begin{aligned}\max\ & a_{01}h_1 + \cdots + a_{0n}h_n,\\ & a_{11}h_1 + \cdots + a_{1n}h_n \geq b_1,\\ & \vdots\\ & a_{m1}h_1 + \cdots + a_{mn}h_n \geq b_m,\\ & h \text{ integral},\end{aligned}$$

and that h' is the only other feasible solution yielding a higher value of the objective function. If h' were not a neighbor of h, then h would – incorrectly – be chosen as the optimal solution for this particular problem.

The set

$$S = \left\{ x \,\middle|\, \sum_j a_{ij}x_j \geq \min\left(\sum_j a_{ij}h_j,\ \sum_j a_{ij}h'_j \right),\ \text{for } i = 0, 1, \ldots, m \right\}$$

is the smallest convex body containing the two lattice points h and h' obtained by varying the right-hand side b. If this set contains some additional lattice points, then h and h' need not be selected as neighbors in the minimal neighborhood system. For if this were so, then in any particular problem for which h is feasible, the test for optimality need not involve h'; the other lattice points in S will do just as well.

Invariance under translation implies that a complete description of the minimal neighborhood system is given by the set of neighbors of the origin. A formal proof of Theorem 2.1 is then obtained by defining, for each lattice point k, the set

$$S_k = \left\{ x \,\middle|\, \sum_j a_{ij}x_j \geq \min\left(0,\ \sum_j a_{ij}k_j \right),\ \text{for } i = 0, 1, \ldots, m \right\},$$

and a partial ordering among nonzero lattice points by $k \lesssim k'$ if and only if $S_k \subseteq S_{k'}$. From Assumption 2.2, each lattice point is preceded by a finite number of other lattice points in this ordering, and there are no pairs for which $k \lesssim k'$ and $k' \lesssim k$. The minimal points in this ordering constitute the minimal neighborhood system for which a local maximum is global.

The minimal neighborhood system is robust under small changes in the technology matrix A as long as Assumption 2.2 is maintained. There may be a sudden discontinuity in the neighborhood system when we pass through a position in which a nonzero lattice point happens to lie on one of the hyperplanes $\sum_j a_{ij}x_j = 0$. Moreover, as Figure 2 illustrates, there may be a certain ambiguity at such a position in the definition of the minimal neighborhood system itself. The ambiguity, though not the discontinuity, may be resolved by adopting a lexicographic tie-breaking rule for coordinates of the vectors $y = Ah$.

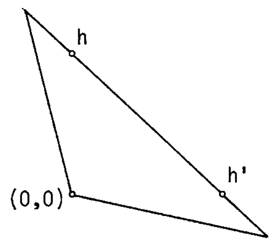

Figure 2. Degeneracy.

3 Complexity of the neighborhood system

Given a description of the minimal neighborhood system, the most immediate algorithm for the solution of integer programming problems is a repeated neighborhood search: A specific feasible solution is proposed and all of its neighbors are examined to see whether one of them is also feasible and yields a higher value of the objective function. If there is such a neighbor, we move to it and iterate; if not, the original feasible vector is optimal.

The amount of computational work associated with this algorithm depends at first glance on the cardinality of the set of neighbors. It is elementary to show that for a technology matrix with three rows and two columns, the minimal neighborhood system will consist of precisely six points. But, aside from this special case, the cardinality of the set of neighbors is not bounded by any function of the number of rows and columns of A and may become arbitrarily large for problems in which m and n are fixed.

The cardinality of the set of neighbors is, however, a crude estimate of the work required by a neighborhood test for optimality. The set of neighbors of the origin may display sufficient structure so that each neighbor need not be examined individually. The minimal neighborhood of the origin may, for example, be the disjoint union of a small number of integral linear segments, as illustrated in Figure 3. Since deciding whether some neighbor in a linear set yields a feasible improvement can be carried out by a small number of divisions, structural regularities in the minimal neighborhood system may lead to a considerable reduction in computational work.

The language of the theory of computational complexity may be used to formulate a general conjecture about minimal neighborhood systems that,

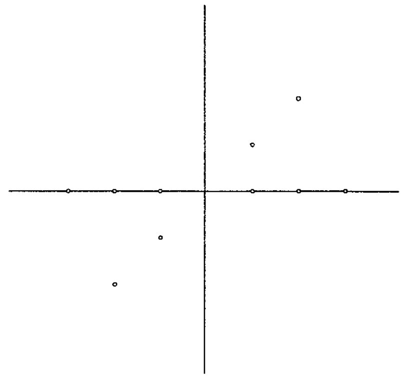

Figure 3. Neighbors of the origin.

if correct, would convert repeated neighborhood searches into an efficient algorithm for integer programming problems. In this theory, the size or complexity of the technology matrix A is defined to be the number of binary bits required to store the entries of A in a digital computer. The number of binary bits required to store an integer a is given by $1+\lceil \log_2(|a|+1)\rceil$, with $\lceil x \rceil$ the smallest integer greater than or equal to x (the additional 1 preceding the logarithm is used to store the sign of a). If the entries in the technology matrix are integers, its size is therefore given by

$$S=\sum_{i=0}^{m}\sum_{j=1}^{n}(1+\lceil \log_2(|a_{ij}|+1)\rceil).$$

An algorithm for the solution of integer programming problems is said to be *polynomial* if the number of steps required for the algorithm to execute successfully is bounded from above by a polynomial function of the size of the problem. Integer programming problems are known to be NP complete, a class of difficult problems for which a polynomial algorithm is extremely unlikely to exist. When the number of variables, or for that matter the number of constraints, is fixed in advance, the class of problems is no longer NP complete and Lenstra (1983) has provided a remarkable algorithm that is indeed polynomial in the size as the remainiong parameters of the technology matrix vary. To the best of my knowledge, however, Lenstra's algorithm has not been tested on a variety of problems, and it may be similar to Katchian's algorithm for linear programming – polynomial in theory, but computationally inefficient.

In a previous publication, I have determined the minimal neighborhood system for a small class of integer programming problems including the transportation problem and the knapsack problem with two activities. In each of these problems, the neighborhood system displays sufficient structure so that a repeated neighborhood search may be accelerated so as to yield a polynomial algorithm. Based on these examples and Lenstra's conclusion, I conjecture that the minimal neighborhood system has a structural description that is polynomial in the data of the problem whenever the number of variables is fixed in advance. Such a description, if possible, may also be capable of economic interpretation in terms of the internal organizational structure of a large economic enterprise whose production possibility set is dominated by significant indivisibilities.

4 Example

In this section, I shall describe a class of technology matrices whose minimal neighborhood systems can be determined by means of a relatively simple algorithm. I do not know, however, whether these neighborhood systems are capable of being described in a polynomial fashion, though I suspect rather strongly that such is the case. The examples show, in a striking fashion, the subtle number-theoretic considerations involved in verifying the general conjecture concerning a polynomial description of the minimal neighborhood system and certainly warrant continued investigation.

One of the basic problems in integer programming is the following: Given a finite set of lattice points in R^n, is there an additional lattice point in their convex hull? I shall consider a specialization that is by no means devoid of interest. The finite set will consist of $n+1$ lattice points, the first n of which are the unit vectors, and the $(n+1)$st point a general integer vector $a=(a_1, a_2, \ldots, a_n)$ all of whose coordinates are strictly positive.

Mathematically, we are concerned with finding an integer vector $(h_1, h_2, \ldots, h_n)$ such that

$$\begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} = \begin{bmatrix} a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & & 1 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_n \end{bmatrix},$$

with $1 > \alpha_j \geq 0$, and $\alpha_0 + \cdots + \alpha_n = 1$. There is a simple algorithm that permits us to determine such a vector if it exists. Adding the n equations, we obtain

$$(h_1+\cdots+h_n)=\alpha_0(a_1+\cdots+a_n)+\alpha_1+\cdots+\alpha_n$$
$$=\alpha_0(a_1+\cdots+a_n)+1-\alpha_0,$$

so that $\alpha_0=h/D$ with $h=h_1+\cdots+h_n-1$ and $D=a_1+\cdots+a_n-1$. But then $h_j=\alpha_j+h\cdot a_j/D$, so that $h_j=\lceil ha_j/D\rceil$. If we define, for integer h between 1 and $D-1$, the function

$$f(h)=\sum_{j=1}^{n}\lceil ha_j/D\rceil,$$

we see that a necessary and sufficient condition for the existence of an additional lattice point in the convex hull is that

$$f(h)=h+1,$$

for some $h=1,2,\ldots,D-1$. The lattice point will be strictly interior to the convex hull if $\alpha_j>0$ for $j=1,2,\ldots,n$, or if ha_j/D is not integral. It will simplify our subsequent analysis if we impose the following condition, which implies that all such lattice points are strictly interior.

Assumption 4.1. For each j, a_j and D are relatively prime.

For example, let $n=3$, and $(a_1,a_2,a_3)=(3,4,5)$, so that $D=11$. The function $f(h)$ is then

h	1	2	3	4	5	6	7	8	9	10
$f(h)$	3	3	5	6	7	8	9	10	12	12

The fact that $f(2)=3$ implies that

$$(1,1,1)=\lceil 3\cdot 2/11\rceil,\ \lceil 4\cdot 2/11\rceil,\ \lceil 5\cdot 2/11\rceil$$

is the convex hull of the three unit vectors and $(3,4,5)$.

The problem has an analytical answer only when $n=2$ or 3. For $n=2$, a necessary and sufficient condition that the triangle with vertices $(1,0)$, $(0,1)$, and (a_1,a_2), with a_1,a_2 positive integers, be free of additional lattice points is simply that $(a_1,a_2)=(1,1)$. (See Fig. 4.)

The solution for $n=3$ provided by Roger Howe a number of years ago is striking. [Proofs of Howe's theorem may be found in Scarf (1985) and Resnick (in press).] A necessary and sufficient condition that the tetrahedron with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and (a_1,a_2,a_3), with a_j positive integers, be free of additional lattice points is that *one* of the

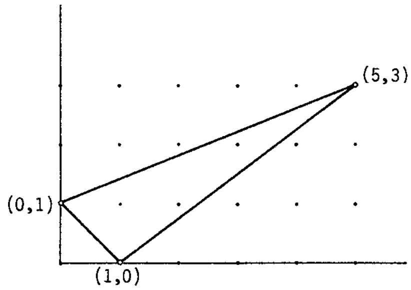

Figure 4. The case $n=2$.

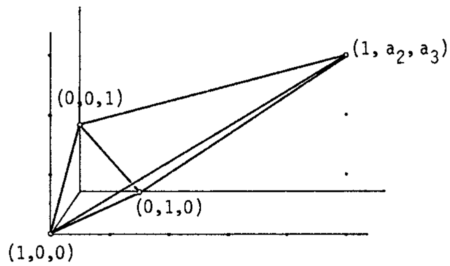

Figure 5. The case $n=3$.

three integers a_1, a_2, a_3 be *unity* and the other two relatively prime. (See Fig. 5.)

When $n>3$, no characterization is presently known for those positive integral vectors $(a_1, \ldots, a_n)$ that, in conjunction with the n unit vectors, give rise to a convex body free of additional lattice points; it seems highly unlikely that there is a simple characterization. The algorithm based on the function $f(h)$ is available for any such vector $(a_1, a_2, \ldots, a_n)$, but it requires $D=-1+\sum a_j$ steps, a quantity that is not polynomial in the size of the problem: $\sum_j (1+\lceil \log_2(a_j+1)\rceil$. It will be instructive to pose the problem as an integer program and to determine its minimal neighborhood system.

The convex hull of our set of $n+1$ lattice points is the intersection of $n+1$ half-spaces, each generated by a hyperplane of dimension $n-1$. The hyperplane passing through the n unit vectors is $x_1+\cdots+x_n=1$; the

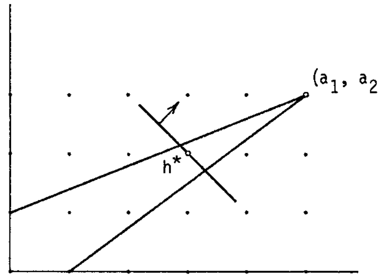

Figure 6. A pair of neighbors.

hyperplane passing through $(a_1, a_2, \ldots, a_n)$ and all of the unit vectors other than the jth unit vector is

$$Dx_j - a_j(x_1 + \cdots + x_n) = -a_j.$$

It follows that the convex hull is defined by the following system of linear inequalities:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ D-a_1 & -a_1 & \cdots & -a_1 \\ -a_2 & D-a_2 & \cdots & -a_2 \\ \vdots & \vdots & \ddots & \vdots \\ -a_n & -a_n & & D-a_n \end{bmatrix} x \geq \begin{pmatrix} 1 \\ -a_1 \\ -a_2 \\ \vdots \\ -a_n \end{pmatrix}.$$

The matrix of coefficients will be denoted by A.

The solution to the integer program

$$\begin{aligned} \max \quad & h_1 + h_2 + \cdots + h_n, \\ & (D-a_1)h_1 - a_1 h_2 - \cdots - a_1 h_n \geq -a_1, \\ & -a_2 h_1 + (D-a_2)h_2 - \cdots - a_2 h_n \geq -a_2, \\ & \vdots \\ & -a_n h_1 - a_n h_2 - \cdots + (D-a_n)h_n \geq -a_n, \\ & h \text{ integral}, \end{aligned}$$

is $(a_1, a_2, \ldots, a_n)$. If the convex hull contains an additional lattice point (let h^* be one that maximizes $h_1 + \cdots + h_n$), then h^* and $(a_1, a_2, \ldots, a_n)$ will be neighbors of each other. (See Fig. 6.)

The entries in the technology matrix A are integers and therefore violate the second part of the nondegeneracy Assumption 2.2. In order to resolve the resulting ambiguity in the definition of the minimal neighborhood system, we adopt the following lexicographic tie-breaking rule:

Lexicographic rule 4.2. Let $y=Ah$ and $y'=Ah'$. For any particular coordinate i, we say that $y_i' >_i y_i$ if the vector $(y_i', y_{i+1}', \ldots, y_0', \ldots, y_{i-1}')$ is lexicographically larger than $(y_i, y_{i+1}, \ldots, y_0, \ldots, y_{i-1})$.

The vector $y=Ah$ will then be a neighbor of the origin if there is no other vector $y'=Ah'$ with

$$\begin{aligned} y_0' &> \min(y_0, 0), \\ y_1' &> \min(y_1, 0), \\ &\vdots \\ y_n' &> \min(y_n, 0), \end{aligned}$$

all of the inequalities being interpreted in this lexicographic sense.

As the following theorem indicates, the neighbors of the origin for the technology matrix A bear an intimate relationship with our algorithm for deciding whether there exists an additional lattice point in the convex hull.

Theorem 4.3. Let $(h_1, h_2, \ldots, h_n)$ be a neighbor of the origin with $h=\sum_1^n h_j > 0$. Then, for each $j=1,2,\ldots,n$,

$$h_j = \lceil a_j h/D \rceil \quad \text{or} \quad \lceil a_j h/D \rceil - 1.$$

We begin the proof of Theorem 4.3 by showing that $-D \le y_i \le D$ for $i=0,1,\ldots,n$, for any neighbor of the origin. Assume, to be specific, that $y_2 > D$. Then

$$A\begin{pmatrix} h_1+1 \\ h_2-1 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1+D \\ y_2-D \\ \vdots \\ y_n \end{pmatrix} > \begin{pmatrix} \min(y_0, 0) \\ \min(y_1, 0) \\ \min(y_2, 0) \\ \vdots \\ \min(y_n, 0) \end{pmatrix},$$

since each coordinate of $(y_0, y_1+D, y_2-D, \ldots, y_n)$, other than the second, is lexicographically greater than the corresponding coordinate of $(y_0, y_1, y_2, \ldots, y_n)$, and $y_2 - D > 0$. Moreover, since $y_2 > D$, $(h_1+1, h_2-1, 0, \ldots, 0)$ is not equal to 0 and is therefore a third vector in the smallest convex set obtained by parallel movements of the inequalities containing h and 0. A similar argument shows that $y_i \le D$ for every i; the inequality

$-D \le y_i$ follows from the fact that the negative of a neighbor of the origin is also a neighbor of the origin.

The inequalities $-D \le y_i \le D$ may be rewritten as

$$-D \le Dh_i - a_i \sum_1^n h_j \le D \quad \text{or} \quad -1 + \frac{a_i h}{D} \le h_i \le 1 + \frac{a_i h}{D},$$

and unless $a_i h/D$ is an integer, it follows that

$$h_i = \lceil a_i h/D \rceil \quad \text{or} \quad \lceil a_i h/D \rceil - 1.$$

Theorem 4.3 tells us that the neighbors of the origin with $\sum h_i \ge 1$ are contained in the set of points obtained by rounding up or down the coordinates of $(a_1, a_2, \ldots, a_n)h/D$, with $h = 1, 2, \ldots, D$. Not all of these points are neighbors of the origin, however. If we write $h_i = \lceil a_i h/D \rceil - \delta_i$ with $\delta_i = 0, 1$, then since $\sum h_i = h$, we must have $\sum_i \delta_i = f(h) - h$. Moreover, if $a_i h = k_i D + r_i$ with $0 < r_i < D$ ($r_i \ne 0$ since a_i and D are relatively prime), then $\lceil a_i h/D \rceil = k_i + 1$, and the vector

$$y = (y_0, y_1, y_2, \ldots, y_n) = (h, (1-\delta_1)D - r_1, (1-\delta_2)D - r_2, \ldots, (1-\delta_n)D - r_n).$$

It follows that

$$\min(0, y_i) = \begin{cases} 0 & \text{if } \delta_i = 0, \\ -r_i & \text{if } \delta_i = 1, \end{cases}$$

for $i = 1, 2, \ldots, n$. The neighbors of the origin with $h \ge 1$ are then obtained by constructing the list of vectors $\min(0, y)$ and eliminating those that are dominated. When this exercise is carried out for our previous example in which $(a_1, a_2, a_3) = (3, 4, 5)$, we obtain the following set of neighbors:

(0, 1, 0)	(1, 1, 0)	(1, 2, 2)
(0, 0, 1)	(1, 1, 1)	(2, 2, 3)
(0, 1, 1)	(1, 2, 1)	(2, 3, 3)
(1, 0, 1)	(1, 1, 2)	(2, 3, 4).

This calculation can be performed quite easily on a computer, but it is, unfortunately, not polynomial in the data of the problem.

A neighbor of the origin with $h = \sum k_j = 0$ will also satisfy $-D \le y_i \le D$. Since $y_i = h_i D - a_i h$ for $i = 1, 2, \ldots, n$, we see that such a neighbor will have $h_i = -1$, 0, or 1, and $y_i = -D$, 0, or D. This observation permits us to characterize – in terms of the minimal neighborhood system – those vectors $(a_1, a_2, \ldots, a_n)$ that, in conjunction with the n unit vectors, generate a convex body free of additional lattice points.

Theorem 4.4. Assume that a_i and D are relatively prime for each i. The convex hull of $(a_1, \ldots, a_n)$ and the n unit vectors will contain no other lattice points if and only if the minimal neighborhood of the origin associated with the matrix A contains some vectors $(h_1, h_2, \ldots, h_n)$ with $\sum h_j = 0$.

The proof of Theorem 4.4 proceeds as follows: If the convex hull contains an additional lattice point, then $f(\hat{h}) = \hat{h}+1$ for some $\hat{h}$. Any neighbor of the origin with $\sum h_j = 0$ will have $y_{i^*} = -D$ for some i^*. If we then define $\hat{h}_i = \lceil a_i \hat{h}/D \rceil - \delta_i$ with $\delta_i = 1$ for $i = i^*$ and $\delta_i = 0$ otherwise, the vector $\hat{y} = A\hat{h}$ will have the form

$$(\hat{h}, D-r_1, D-r_2, \ldots, -r_{i^*}, \ldots, D-r_n)$$

and will be strictly larger, in all coordinates, than $\min(0, y)$. It follows that there are no neighbors of the orgin with $\sum h_j = 0$.

On the other hand, if the convex hull of $(a_1, a_2, \ldots, a_n)$ and the n unit vectors contains no other lattice points, it will be true that $f(h) \geq h+2$ for all h. From Theorem 4.3, we see that any neighbor of the origin $\xi = (\xi_0, \xi_1, \ldots, \xi_n)$, with $\xi_0 \geq 1$, will have at least *two* coordinates ξ_i and ξ_j that are strictly negative. But if $h = (1, -1, 0, \ldots, 0)$, then

$$y = Ah = (0, D, -D, 0, \ldots, 0);$$

$\min(0, y)$ cannot be dominated by any neighbor of the origin, and y must therefore be a neighbor of the origin itself. This demonstrates Theorem 4.4.

REFERENCES

Arrow, K. J. (1951), "An extension of the basic theorems of classical welfare economies," in J. Neyman (Ed.), *Proceedings of the second Berkeley symposium on mathematical statistics and probability,* Berkeley and Los Angeles: University of California Press, pp. 507-32.

Arrow, K. J. (1983), *Collected papers: General equilibrium,* Cambridge, Mass.: Harvard University Press.

Arrow, K. J. and L. Hurwicz (1958), "On the stability of the competitive equilibrium, pt. 1," *Econometrica,* 26: 522-52.

Arrow, K. J., H. D. Block, and L. Hurwicz (1959), "On the stability of the competitive equilibrium, pt. 2," *Econometrica,* 27: 82-109.

Debreu, G. (1951), "The coefficient of resource utilization," *Econometrica,* 19: 273-92.

Lenstra, H. W., Jr. (1983), "Integer programming with a fixed number of variables," *Mathematics of Operations Research,* 8: 538-48.

Lerner, A. (1944), *The economics of control,* New York: Macmillan.

Resnick, B. (in press), "Lattice point simplices," *Discrete Mathematics*.
Samuelson, P. A. (1941–42), "The stability of equilibrium," *Econometrica,* 9: 97–120; 10: 1–25.
Scarf, H. E. (1981), "Production sets with indivisibilities," *Econometrica,* 49: 1–32; 395–423.
Scarf, H. E. (1985), "Integral polyhedra in three space," *Mathematics of Operations Research,* 10(3): 403–38.
Scarf, H. E. (in press), "Neighborhood systems for production sets with indivisibilities," *Econometrica*.